

Photomultiplier tube spectra



Milind Diwan
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BNL physics department
summer lectures.

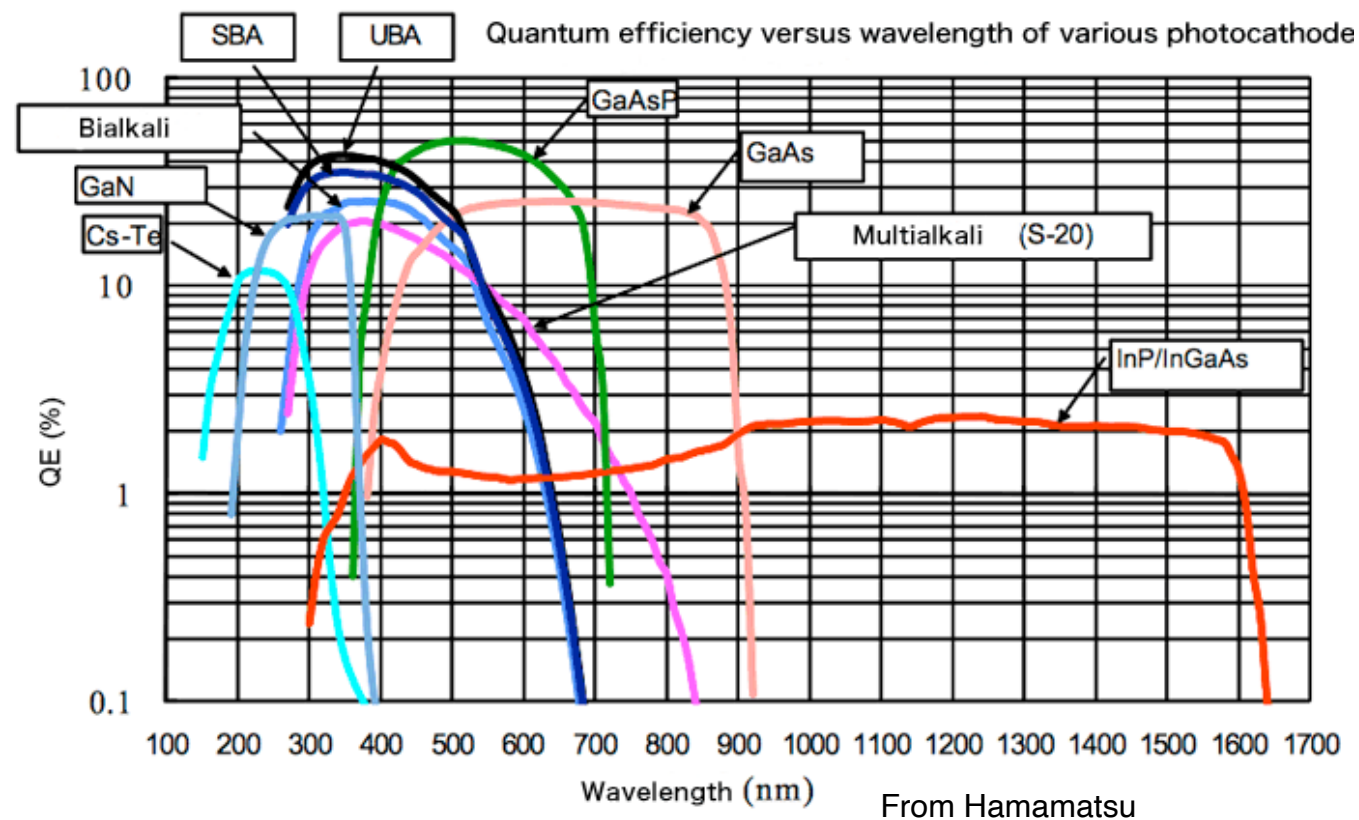
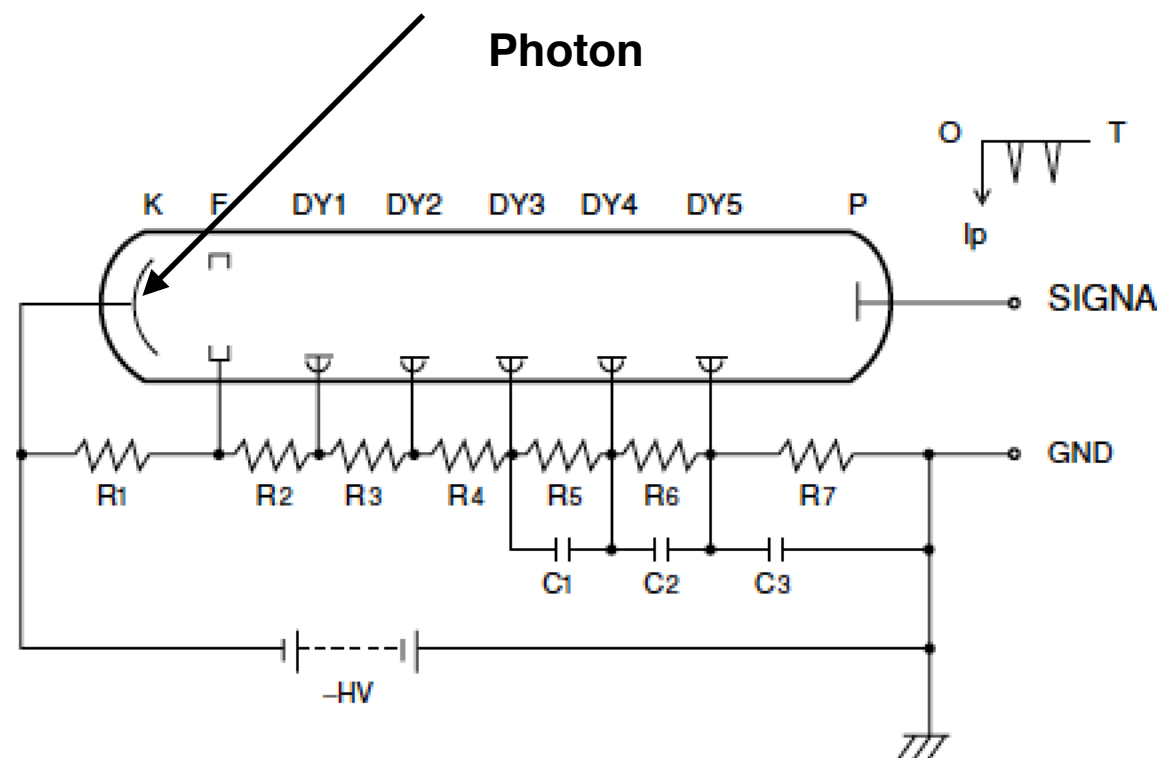
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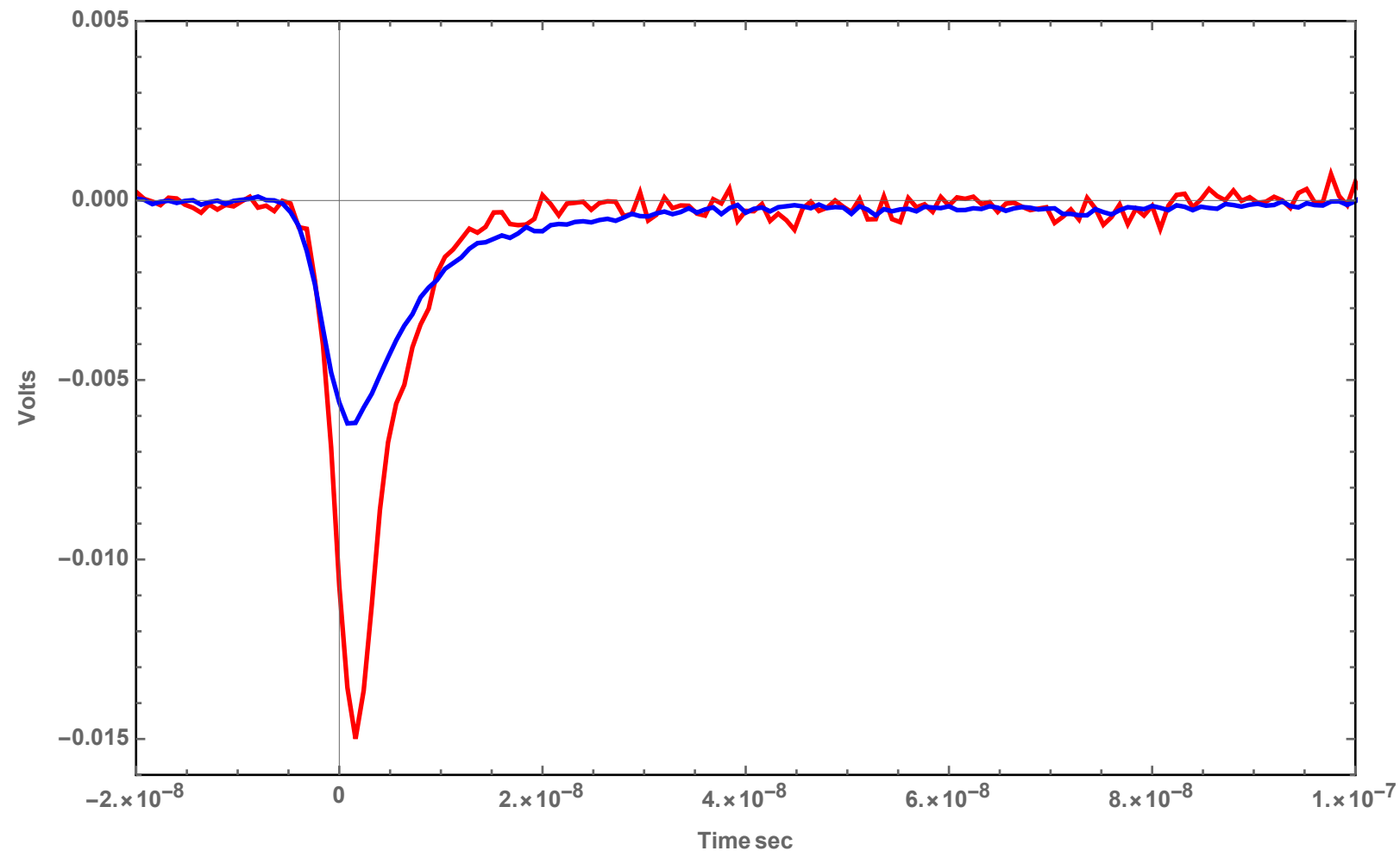
Outline

- *In this lecture we are going to do three things*
 - *We will learn the basics of how a photomultiplier tube works. This is a very important sensor. The electrical signal out of a PMT has been analyzed in a separate lectures.*
 - *In these set of slides we will derive the expected charge spectra for photomultiplier tubes. The techniques can be applied to other detectors as well.*
 - *To derive these we will need some powerful mathematical tools. We will get an introduction to these tools.*
- *The assumption in these slides is that we have integrated the charge coming from a PMT. The only sources of noise is the PMT itself.*
- *Reference: See the guide from Hamamatsu, Photomultiplier tubes: Basics and Applications (2007) 3rd edition. There are many papers in journals such as Nuclear Instruments and Methods.*

Photo-Multiplier Tube



- The photoelectric (Hertz 1887!) causes metals to eject electrons in response to light.
- Einstein's Nobel prize was for explaining the photoelectric effect with quantum mechanics !
- Electric fields accelerate and multiply the primary electron in several stages. Each stage has multiplication of $\sim 4-5$.
- Typical Gain = $AV^{kn} \sim 10^6 - 10^7$ where V is the typical voltage \sim few 1000 V.
- Time resolution < 10 ns.
- Transit time can be < 1 microsec
- PMT first stage is sensitive to small magnetic fields.
- Many clever geometries.



These are a couple of typical pulses.
We are going to integrate pulses like
these to get the total charge.

R has two grandparents. Each is going to give R some money for her birthday. \$10, \$20, \$30, or \$40 with probability of 0.1, 0.2, 0.3, 0.4. What is the probability that R gets a total of \$50 ?

How do we make a method that provides an answer quickly ? It is called a characteristic function. It keeps track of numbers. Gauss described this as a clothes-line to find the number that you need.

$$\phi(s)_R = 0.1e^{10s} + 0.2e^{20s} + 0.3e^{30s} + 0.4e^{40s}$$

To get the total you multiply the char. functions.

$$\phi(s)_{R1+R2} = (0.1e^{10s} + 0.2e^{20s} + 0.3e^{30s} + 0.4e^{40s})^2$$

$$\phi(s)_{R1+R2} = 0.01e^{20s} + 0.04e^{30s} + 0.1e^{40s} + 0.2e^{50s} + 0.25e^{60s} + 0.24e^{70s} + 0.16e^{80s}$$

**Notice that you can just read off the probability that R gets \$50
The characteristic function always exists and encodes all
information about a probability density**

Some definitions

X is a continuous random variable with probability density function $P_X(x)$ then the characteristic function is

$$\varphi_X(k) \equiv E[e^{ikx}] = \int_{-\infty}^{\infty} P_X(x) e^{ikx} dx$$

X is a random variable, x is a realization of X over its domain. $E[\]$ is the expectation value of its argument. What is the point of the characteristic function ?

It is a way of tagging the probability number with a unique number (e^{ikx}). It is as if we are storing the number in a file folder with a tag.

Notice $\varphi(k=0) = 1$ since it is the integral of the PDF.

the mean is given by $\langle x \rangle = -i \frac{\partial \varphi}{\partial k}(k=0)$, and

Higher derivatives lead to higher moments....

If X and Y are two random variables and $z = f(x,y)$ then the Characteristic function for Z is

$$\varphi_Z(k) = \iint e^{ikf(x,y)} P_X(x) dx Q_Y(y) dy$$

To get the moments of $f(x,y)$ often it is not necessary to evaluate the integral.

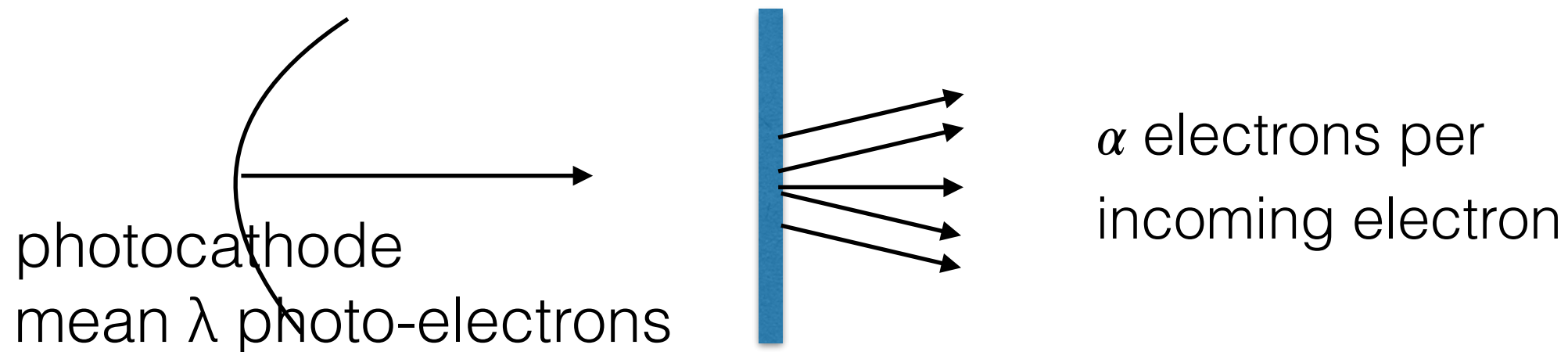
e.g. $f(x,y) = x + y \Rightarrow \varphi_Z(k) = \varphi_X(k) \cdot \varphi_Y(k)$ leave it for you to prove this

basics of photomultiplier

Mean of λ photo electrons come from the photo-cathode to the first dynode.

Each electron generates α electrons at the first dynode.

Each subsequent stage produces gain of few electrons per incoming electrons leading to gains of $\sim 10^6$ - 7



If mean of λ photons convert in a photo-sensor with a mean gain of α electrons per photon what is the distribution of the output number of electrons ?

Basically, an average of λ packets arrive each with an average of α items in each packet. What is the mean and variance of the total number of items ? How do we calculate this...

Poisson probability mass function and its Char. function

Distribution of incoming electrons with Poisson mean of λ

K is the number of electrons, a discrete random variable with probability mass function

$$P_K(k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Characteristic function for this is the expectation value of e^{isk}

$$\varphi_K(s) \equiv E[e^{isk}] = \sum_{k=0}^{\infty} e^{isk} P_K(k) = \sum_{k=0}^{\infty} \frac{e^{isk} \lambda^k}{k!} e^{-\lambda} = e^{e^{is} \lambda} e^{-\lambda} = e^{\lambda(e^{is}-1)}$$

This function has many interesting properties.

secondaries

Similarly, the distribution of from each electron has probability

$$P_L(\ell) = \frac{\alpha^\ell}{\ell!} e^{-\alpha} \quad \text{and a similar characteristic function } \varphi_L(s).$$

We have labeled the two functions to distinguish them from each other.

Additionally recall that the mean for a Poisson distribution with parameter λ is

$$E[k] = \lambda \quad \text{and the variance is also } E[k^2] - (E[k])^2 = \lambda$$

Debatable points

- ***Why should the number of ejected electrons from the photo-cathode be Poisson distributed ?***
 - ***Recall that only a small fraction of total photons from a scintillation event are caught by the PMT, and then with an efficiency of 20-30% converted to electrons. And so this is a Poisson/Binary process. The result is a Poisson distribution for the photo-electrons.***
- ***Why should the emission from the first dynode be Poisson distributed ?***
 - ***If the dynode is uniform then only a small fraction of total excitations from a penetrating electron results in an emitted electron. However, there could be many geometrical reasons why the dynode response is not uniform. For exampl, an electron could miss a dynode entirely.***
- ***And so both of these are reasonable assumptions for well performing detectors in normal situations.***

Simple calculation first

Before we do the full calculation we will perform an intuitive calculation.

Obviously λ and α are Poisson parameters. Total charge will be called Q.

$$\langle Q \rangle = \lambda \cdot \alpha$$

The fractional variance of Q will have contribution from the fluctuation of the incident number K and

then the fluctuation in the secondary number $\sum_{i=1}^K L_i$ which is a sum of K random numbers each Poisson

distributed with parameter α

$$\frac{\text{Var}[Q]}{\langle Q \rangle^2} = \frac{\text{Var}[K]}{\langle K \rangle^2} + \frac{1}{K} \times \frac{\text{Var}[L]}{\langle L \rangle^2} = \frac{1}{\lambda} + \frac{1}{\lambda} \cdot \frac{1}{\alpha}$$

$$\text{Var}[Q] = \lambda \alpha (\alpha + 1)$$

Suppose L is actually Normally distributed with parameters α (mean) and σ (standard deviation)

$$\frac{\text{Var}[Q]}{\langle Q \rangle^2} = \frac{\text{Var}[K]}{\langle K \rangle^2} + \frac{1}{K} \times \frac{\text{Var}[L]}{\langle L \rangle^2} = \frac{1}{\lambda} + \frac{1}{\lambda} \cdot \frac{\sigma^2}{\alpha^2}$$

$$\text{Var}[Q] = \lambda (\sigma^2 + \alpha^2)$$

Now calculate the PDF for the total charge

Total charge is a discrete random number Z .

$$Z = \sum_{i=1}^K L_i$$

This is a sum of K random numbers, each is the count from an electron gain.

Now, it is obvious that the mean number of total electrons must be $\lambda \times \alpha$, where λ is the Poisson mean for the number of electrons and α is the Poisson mean for the gain or the number of electrons resulting from the multiplication of an electron.

However, the random number for the total number of photons is not a product of the two random numbers K (the number of electrons) and L (the number of photons).

The characteristic function for the number Z is

$$\varphi_Z(s) \equiv E[e^{isz}] = \sum_{z=0}^{\infty} e^{isz} P_Z(Z = z)$$

Here P_Z is unknown.

Characteristic function of Z

Start with the generating function for total number Z

(recall that K is the r.v. for electrons and L is the r.v. for gain on each electron)

$$\begin{aligned}\varphi_Z(s) &\equiv \sum_{z=0}^{\infty} e^{isz} P_Z(Z=z) = E[e^{isz}] = \sum_{k=0}^{\infty} E[e^{is \sum_{i=1}^K L_i} | K=k] \cdot P_K(k) \\ &= \sum_{k=0}^{\infty} E[e^{isl_1} e^{isl_2} \dots e^{isl_K} | K=k] \cdot P_K(k)\end{aligned}$$

This says that the total expectation for e^{isz} is the same as the average of the conditional expectation for k electrons (averaged over the probability of obtaining k electrons). This is the law of total expectation. Each random variable L_i is independent, and so each has the same char. func.

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (\varphi_L(s))^k \cdot P_K(k)$$

This leads to a compact expression.

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (e^{\alpha(e^{is}-1)})^k \cdot \frac{e^{-\lambda} \lambda^k}{k!} = e^{\lambda(e^{\alpha(e^{is}-1)}-1)}$$

expressions for Poisson and Normal gain

For Poisson gain

$$\varphi_L(s) = e^{\alpha(e^{is}-1)}$$

For gain with normal PDF. $N(\mu, \sigma^2)$

$$\varphi_L(s) = e^{is\mu} \cdot e^{-\frac{s^2\sigma^2}{2}}$$

This is the C.F. for a normal PDF.

Generally for Poisson PDF with $\alpha \gg 1$ we can use Normal PDF with $\mu=\alpha$ and $\sigma^2 = \alpha$

For Poisson

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (e^{\alpha(e^{is}-1)})^k \cdot \frac{e^{-\lambda} \lambda^k}{k!} = \sum_{k=0}^{\infty} e^{k\alpha(e^{is}-1)} \cdot \frac{e^{-\lambda} \lambda^k}{k!}$$

When we invert this to get the probability we get a formula with an infinite series

$$P_Z(n) = \begin{aligned} & e^{-\lambda} + \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \cdot e^{-k\alpha} && \text{for } n = 0 \\ & \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \cdot \frac{e^{-k\alpha} (k\alpha)^n}{n!} && \text{for } n > 0 \end{aligned}$$

It is important to be careful about 0

This has many names: compound Poisson, jumping Poisson, etc.

Normal distributed gain

For Normal

$$\varphi_Z(s) = \sum_{k=0}^{\infty} (e^{is\mu} \cdot e^{-\frac{s^2\sigma^2}{2}})^k \cdot \frac{e^{-\lambda} \lambda^k}{k!} = \sum_{k=0}^{\infty} (e^{is\mu k} \cdot e^{-\frac{s^2\sigma^2 k}{2}}) \cdot \frac{e^{-\lambda} \lambda^k}{k!} = e^{\lambda [e^{is\mu - s^2\sigma^2/2} - 1]}$$

When we invert this to get the probability we get a formula with an infinite series

$$P_Z(z) = \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \cdot \frac{1}{\sqrt{2\pi\sigma^2 k}} e^{-\frac{(z-k\mu)^2}{2\sigma^2 k}} \quad \text{for } z > 0$$

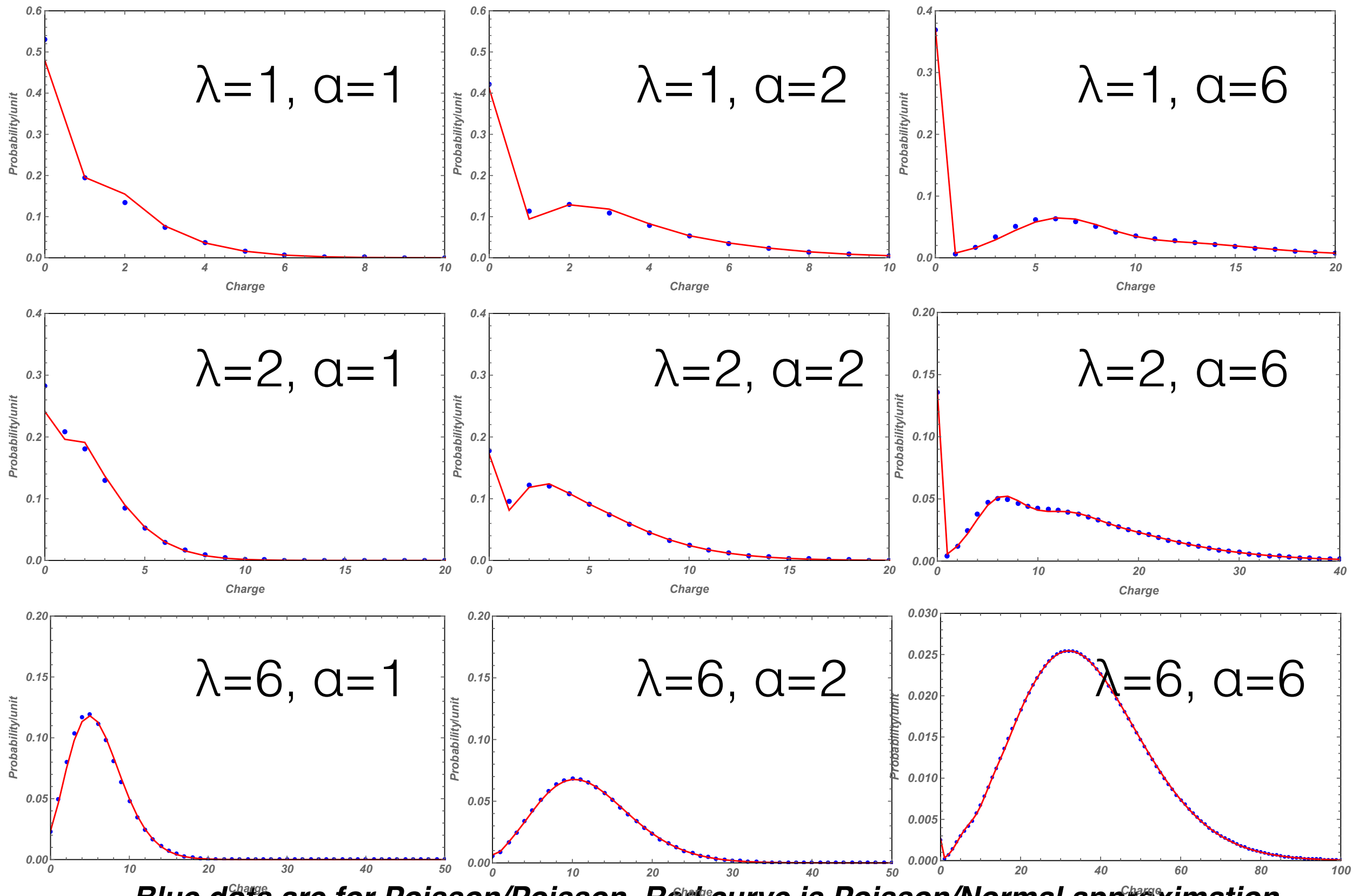
There is some care needed here because z is now a continuous variable, and $P_Z(z)$ is now a probability density function.

(When the bin size of z is chosen to be $\delta z = 1$, we recover an approximation to the probability mass function for a Poisson jumping distribution.)

One still needs to obtain a well defined probability at $z = 0$. For this we go back to setting $\mu = \alpha$ and $\sigma^2 = \alpha$

$$P_Z(n) \approx \begin{aligned} & e^{-\lambda} + \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \cdot \frac{e^{-k\alpha/2}}{\sqrt{2\pi k\alpha}} \quad \text{for } n = 0 \\ & \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \cdot \frac{1}{\sqrt{2\pi k\alpha}} e^{-\frac{(n-k\alpha)^2}{2k\alpha}} \quad \text{for } n > 0 \end{aligned}$$

comparison of plots



***Blue dots are for Poisson/Poisson, Red curve is Poisson/Normal approximation
 $\lambda = \{1, 2, 6\}$ is the Poisson parameter; $\alpha = \{1, 2, 6\}$ is the Poisson parameter for the jump***

pedestal and background

The signal is going to be convoluted with background processes. First we have to figure out what the background looks like in the absense of signal.

We take an integral (or sum) in a given time interval over which signal might arrive. If there is no signal then we end up adding a fluctuating baseline with some mean. This will have a Gaussian PDF

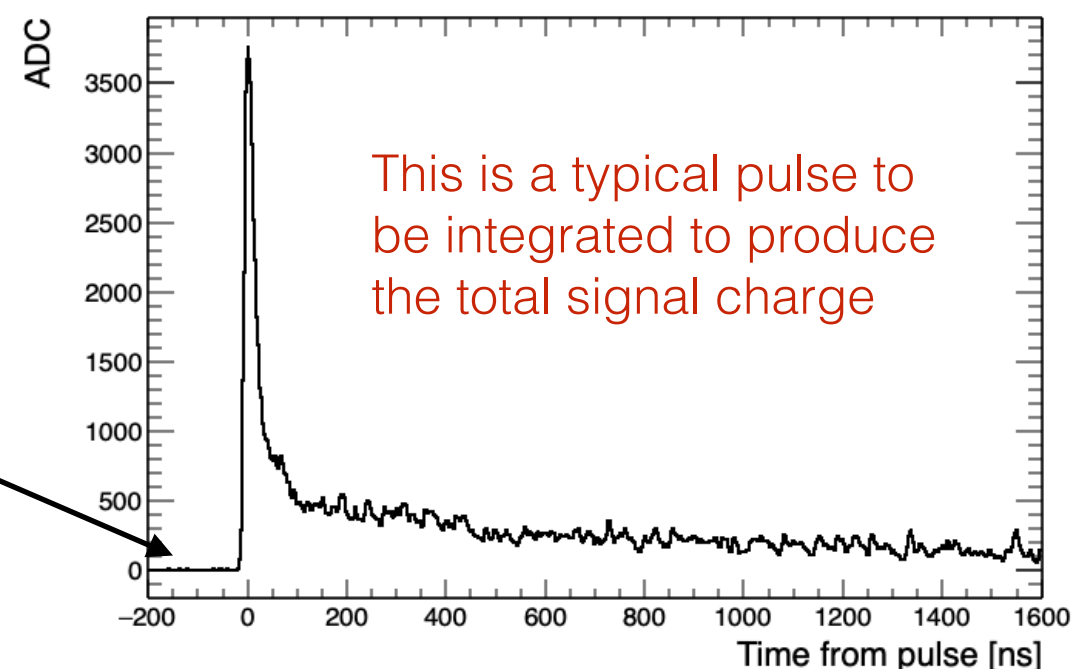
$$P_Q(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(x-q_0)^2}{2\sigma_0^2}} \quad \text{where } q_0 \text{ is the pedestal}$$

There is another background process due to emission of real electrons from either the photocathode or one of dynodes due to thermal fluctuations. This will have an exponential PDF, but with some chance that there is no emission at all.

$$P_D(x) = (1-w)\delta(x) + w\theta(x)c_0e^{-c_0x}$$

here w is the probability of emission, $\delta(x)$ is the dirac delta function to create a generalized distribution, and $\theta(x)$ is a step function, and c_0 is some constant.

The baseline fluctuates producing noise. When this is summed it will contribute a Gaussian distributed charge. The baseline could be shifted and has a fluctuation.



background function

We first have to convolute P_Q and P_D to get the background only spectrum for $B = Q + D$; We can do this explicitly or just write down the characteristic function.

$$\varphi_B(s) = \varphi_Q(s) \cdot \varphi_D(s) = e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2} \times \left((1-w) + w \frac{1}{1 - is/c_0} \right)$$

as long as the width of the pedestal is small with respect to the exponential $1/c_0$

$$P_B(x) = (1-w) \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(x-q_0)^2}{2\sigma_0^2}} + w \theta(x-q_0) c_0 e^{-c_0(x-q_0)}$$

Basically there is a pedestal with some width and a falling exponential background.

What if the pedestal width is too wide ? Then the exponential part of the background will get consumed in the width of the pedestal.

It is possible to obtain the full form.

C.F. for an exponential has the form $1/(a-is)$

Gaussian-modified-exponential

A normally distributed random number with an addition of an exponential random number is called an exponential-Gaussian or Gaussian-exponential.

The characteristic function is

$$\varphi_{EG}(s) = \frac{e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2}}{(1 - is / c_0)}$$

The PDF that corresponds to this is

$$P_{EG}(x) = \frac{c_0}{2} e^{\frac{c_0^2 \sigma_0^2}{2}} e^{-c_0(x-q_0)} \text{Erfc} \left[\frac{1}{\sqrt{2}} \left(c_0 \sigma_0 - \frac{x-q_0}{\sigma_0} \right) \right]$$

Recall that c_0 is the constant for dark rate, σ_0 is the std. dev. of the pedestal and q_0 is the pedestal.

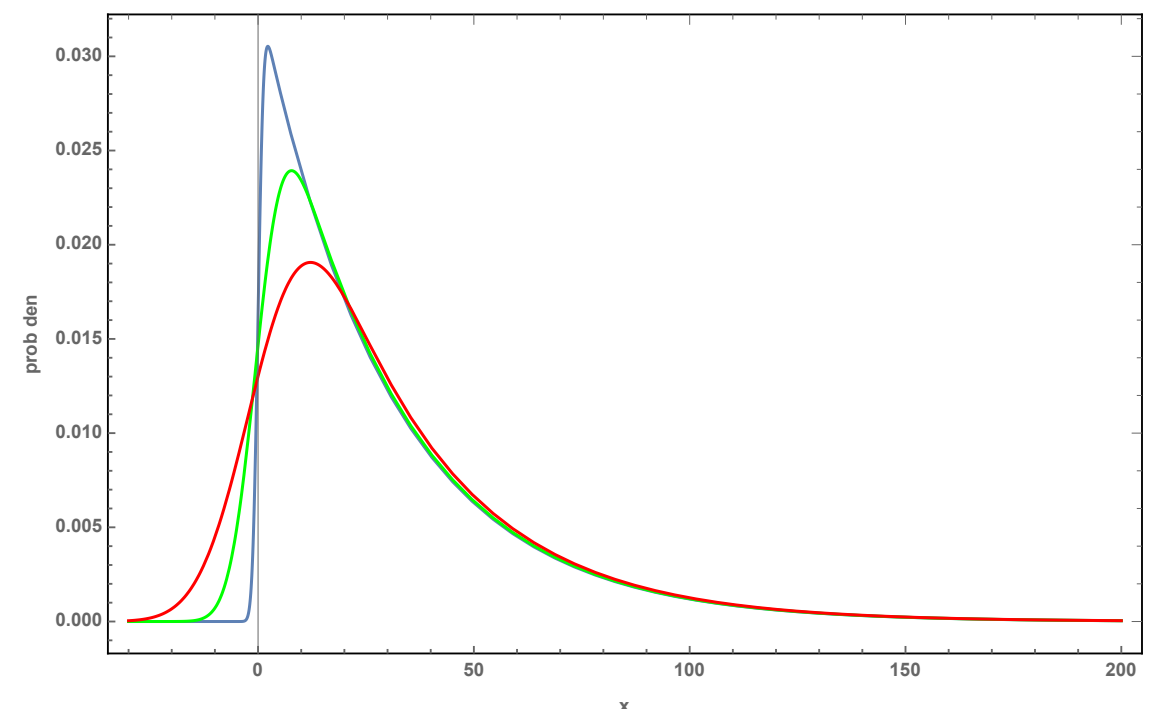
$\text{Erfc}[x]$ is the complement of the error function.

$$\text{Erfc}[x] = 1 - \text{Erf}[x] = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

When $c_0 \sigma_0$ is small the Erfc acts like a step function. Some care is needed in calculation in case of negative or very large arguments.

The Mean of the PDF is $(q_0 + 1 / c_0)$

The Variance is $(\sigma_0^2 + 1 / c_0^2)$



Couple of definitions

Let's define some PDFs to get a compact expression

The Normal PDF

$$N(x : \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Exponential modified Normal PDF

$$E_N(x : \mu, \sigma^2, \lambda) = \frac{\lambda}{2} e^{\frac{\lambda^2 \sigma^2}{2}} e^{-\lambda(x-\mu)} \operatorname{Erfc} \left[\frac{1}{\sqrt{2}} \left(\lambda \sigma - \frac{x-\mu}{\sigma} \right) \right]$$

μ : mean of the Gaussian

σ^2 : variance of the Gaussian

λ : exponential decay parameter

total response

To get the complete response we have to get the PDF for $Y = B + Z$ where B is the background and Z is the signal. $B = D + Q$ as we calculated for the background. and so

$$\varphi_Y(s) = \varphi_Z(s) \cdot \varphi_D(s) \cdot \varphi_Q(s)$$

$$\varphi_Y(s) = \left(\sum_{k=0}^{\infty} (e^{is\mu k} \cdot e^{-\frac{s^2\sigma^2 k}{2}}) \cdot \frac{e^{-\lambda} \lambda^k}{k!} \right) \times \left((1-w) + w \frac{1}{1-is/c_0} \right) \times e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2}$$

This is the full and complete expression for the PMT response assuming the gain is normally distributed. We will break this up in 6 pieces and analyze it for special conditions.

$$\varphi_Y(s) = \left(\sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} (e^{is(\mu k + q_0)} \cdot e^{-\frac{s^2(\sigma^2 k + \sigma_0^2)}{2}}) \right) \times \left((1-w) + w \frac{1}{1-is/c_0} \right) \times e^{isq_0} e^{-\frac{1}{2}\sigma_0^2 s^2}$$

We break this up in three cases for p.e. count: $k = 0$, $k = 1$, and $k > 1$

And additional two cases for $(1-w)$, without dark rate

and (w) , with dark rate addition.

all terms broken out for the characteristic function. Check that when $s=0$ the sum adds to 1

	<i>Terms</i>	$(1-w) \times$	$w \times$
<i>no signal</i>	$e^{-\lambda} \times$	$e^{isq_0} e^{-\frac{s^2 \sigma_0^2}{2}}$	$e^{isq_0} e^{-\frac{s^2 \sigma_0^2}{2}} \times \frac{1}{1-is/c_0}$
<i>single pe</i>	$\lambda e^{-\lambda} \times$	$e^{is(\mu+q_0)} e^{-\frac{s^2(\sigma^2+\sigma_0^2)}{2}}$	$e^{is(\mu+q_0)} e^{-\frac{s^2(\sigma^2+\sigma_0^2)}{2}} \times \frac{1}{1-is/c_0}$
<i>many pe</i>	$\sum_{k=2}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \times$	$e^{is(\mu k+q_0)} e^{-\frac{s^2(\sigma^2 k+\sigma_0^2)}{2}}$	$e^{is(\mu k+q_0)} e^{-\frac{s^2(\sigma^2 k+\sigma_0^2)}{2}} \frac{1}{1-is/c_0}$
		<i>no dark current</i>	<i>with dark current</i>

all these are to be added together. When transformed to PDF, each term will convert to a normal PDF or an exponential-normal PDF.

Approximation when pedestal is narrow, also set pedestal $q_0 = 0$

	<i>Terms</i>	$(1 - w) \times$	$w \times$
<i>no signal</i>	$e^{-\lambda} \times$	$e^{-\frac{s^2 \sigma_0^2}{2}}$	$\frac{1}{1 - is / c_0}$
<i>single pe</i>	$\lambda e^{-\lambda} \times$	$e^{is(\mu)} e^{-\frac{s^2(\sigma^2)}{2}}$	$e^{is(\mu)} e^{-\frac{s^2(\sigma^2)}{2}} \times \frac{1}{1 - is / c_0}$
<i>many pe</i>	$\sum_{k=2}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \times$	$e^{is(\mu k)} e^{-\frac{s^2(\sigma^2 k)}{2}}$	$e^{is(\mu k)} e^{-\frac{s^2(\sigma^2 k)}{2}} \frac{1}{1 - is / c_0}$
		<i>no dark current</i>	<i>with dark current</i>

$\sigma_0 \ll \sigma$ and $\sigma_0 \ll 1 / c_0$
also set $q_0 = 0$

when $w=0$ or c_0 is very small

no signal

$$\begin{array}{cc} \text{Terms} & (1-w) \times \\ e^{-\lambda} \times & e^{-\frac{s^2 \sigma_0^2}{2}} \end{array}$$

single pe

$$\lambda e^{-\lambda} \times e^{is(\mu)} e^{-\frac{s^2(\sigma^2)}{2}}$$

many pe

$$\sum_{k=2}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \times e^{is(\mu k)} e^{-\frac{s^2(\sigma^2 k)}{2}}$$

*no dark
current*

~~$$\begin{array}{cc} w \times & \\ \frac{1}{1-is/c_0} & \\ e^{is(\mu)} e^{-\frac{s^2(\sigma^2)}{2}} \times \frac{1}{1-is/c_0} & \\ e^{is(\mu k)} e^{-\frac{s^2(\sigma^2 k)}{2}} \frac{1}{1-is/c_0} & \end{array}$$~~

*with dark
current*

$$\sigma_0 \ll \sigma \text{ and } \sigma_0 \ll 1/c_0$$

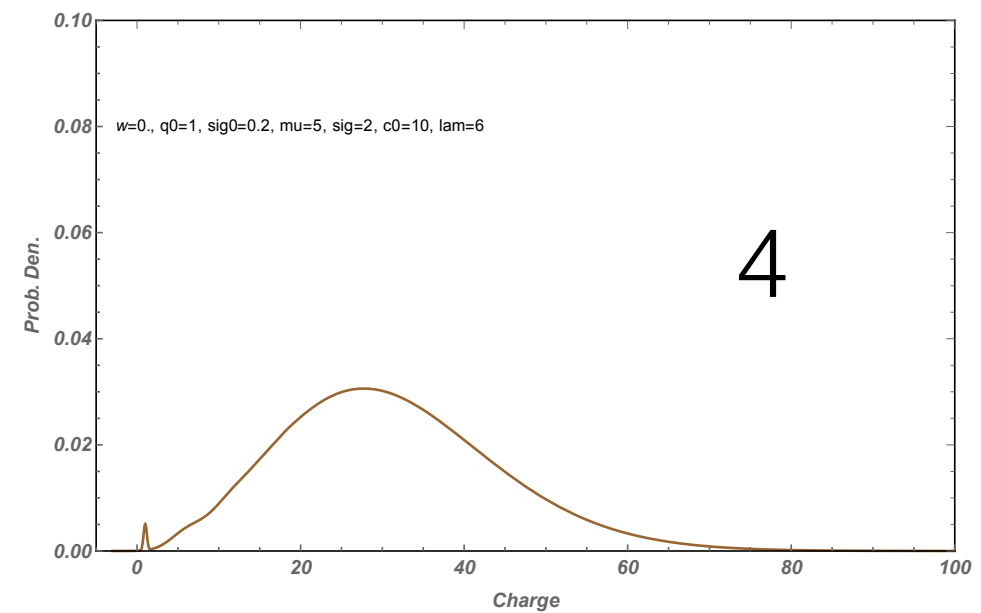
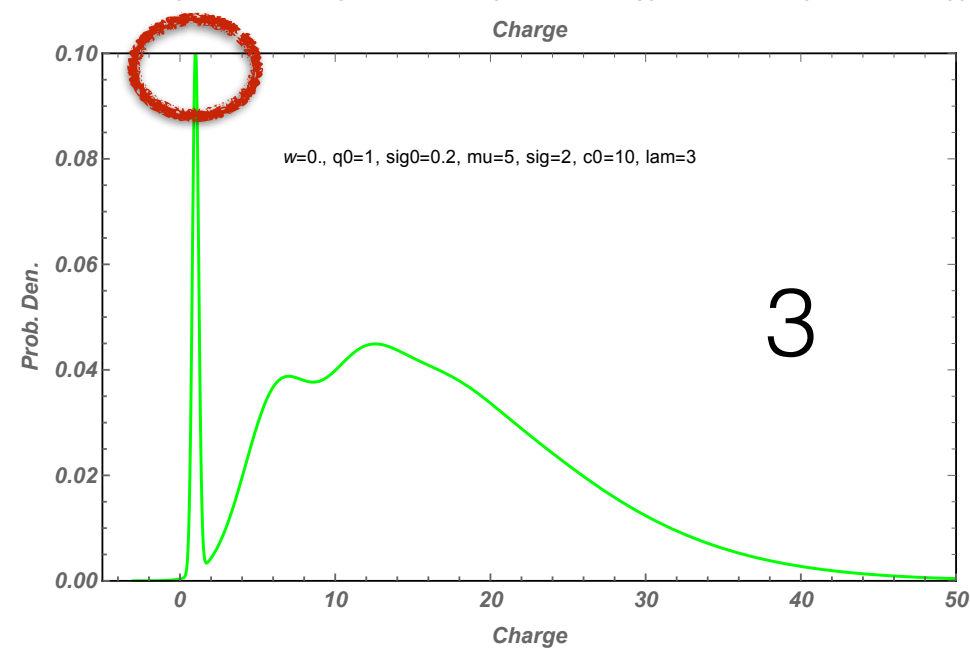
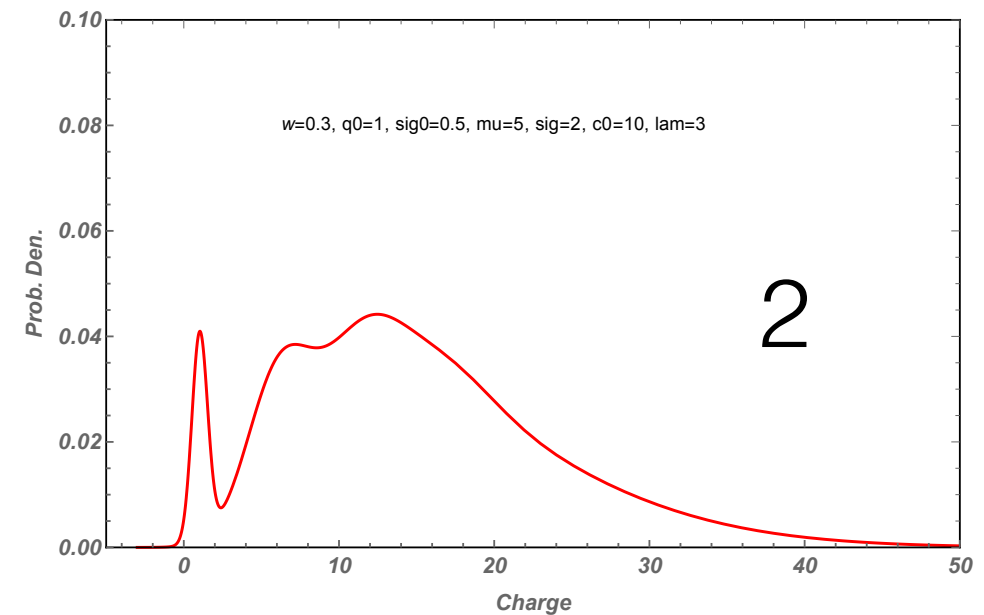
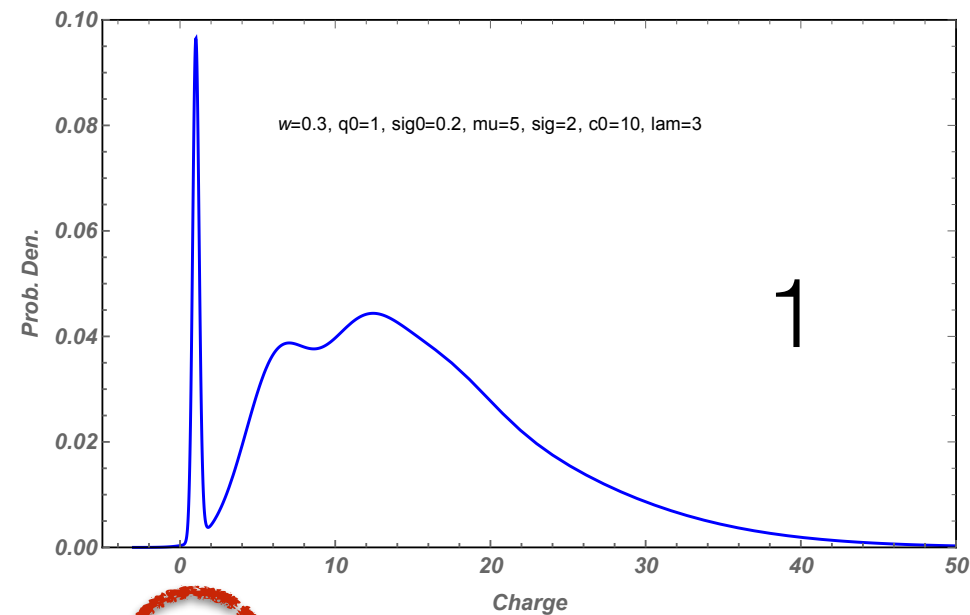
$$\text{also set } q_0 = 0$$

all terms broken out for the PDF in compact notation

	<i>Terms</i>	$(1-w) \times$	$w \times$
<i>no signal</i>	$e^{-\lambda} \times$	$N(x : q_0, \sigma_0^2)$	$E_N(x : q_0, \sigma_0^2, c_0)$
<i>single pe</i>	$\lambda e^{-\lambda} \times$	$N(x : \mu + q_0, \sigma^2 + \sigma_0^2)$	$E_N(x : \mu + q_0, \sigma^2 + \sigma_0^2, c_0)$
<i>many pe</i>	$\sum_{k=2}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} \times$	$N(x : \mu k + q_0, \sigma^2 k + \sigma_0^2)$	$E_N(x : \mu k + q_0, \sigma^2 k + \sigma_0^2, c_0)$
		<i>no dark current</i>	<i>with dark current</i>

***all these are to be added together to get the full PDF.
The sum is applied across the row.***

plot some examples

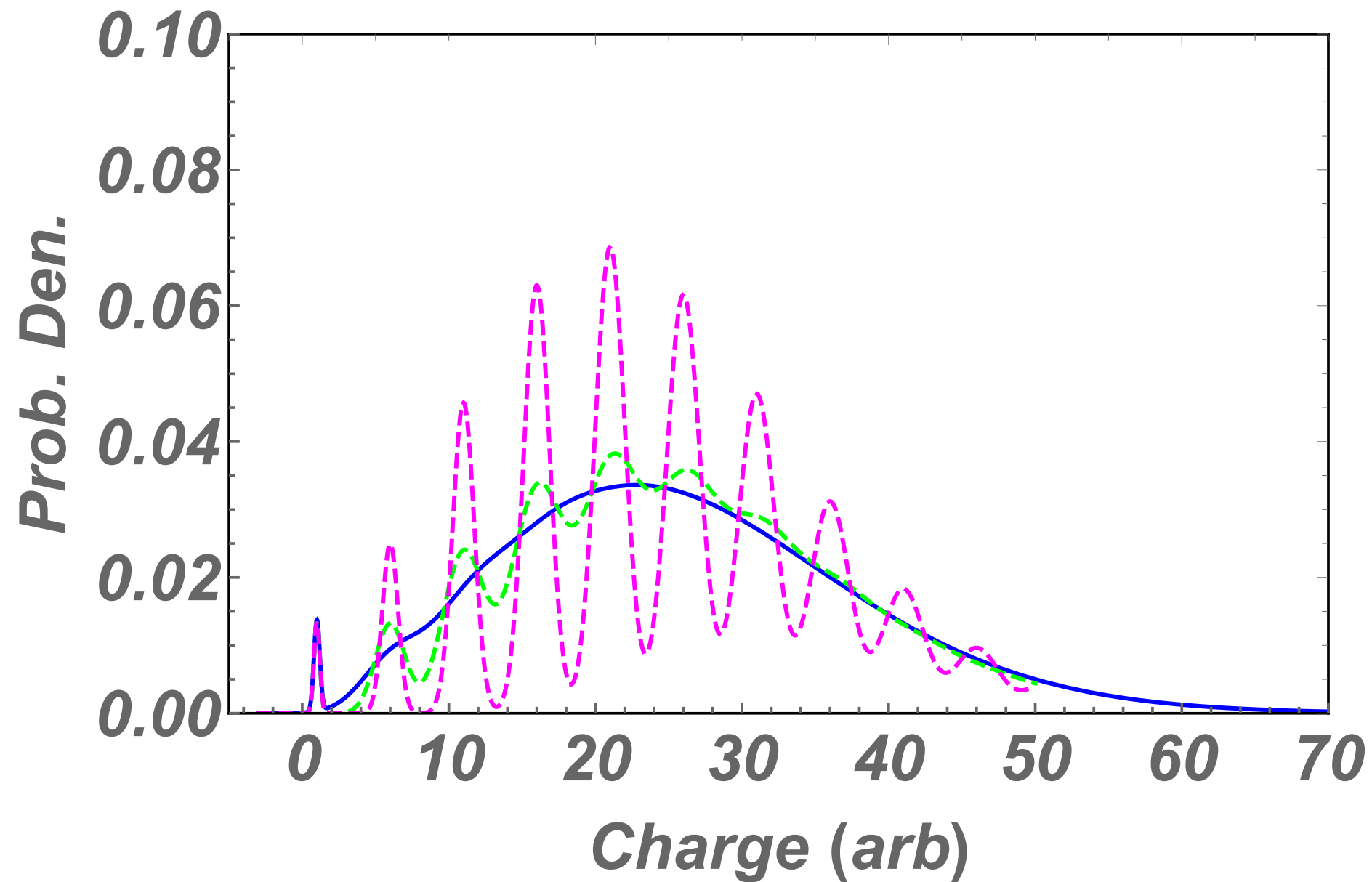


<i>plot</i>	λ	w	q_0	σ_0	μ	σ	c_0
1	3	0.3	1	0.2	5	2	10
2	3	0.3	1	0.5	5	2	10
3	3	0	1	0.2	5	2	10
4	6	0	1	0.2	5	2	10

with dark current, narrow/wide pedestal

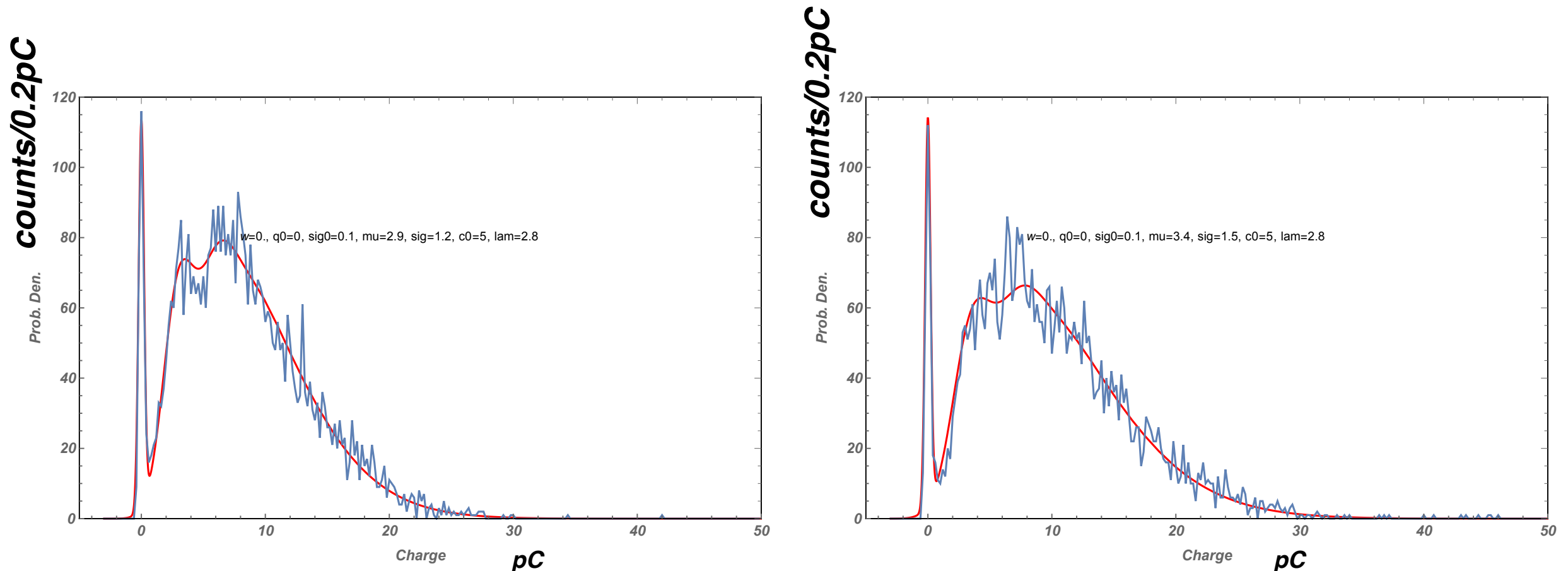
no dark current, less/more p.e.

example



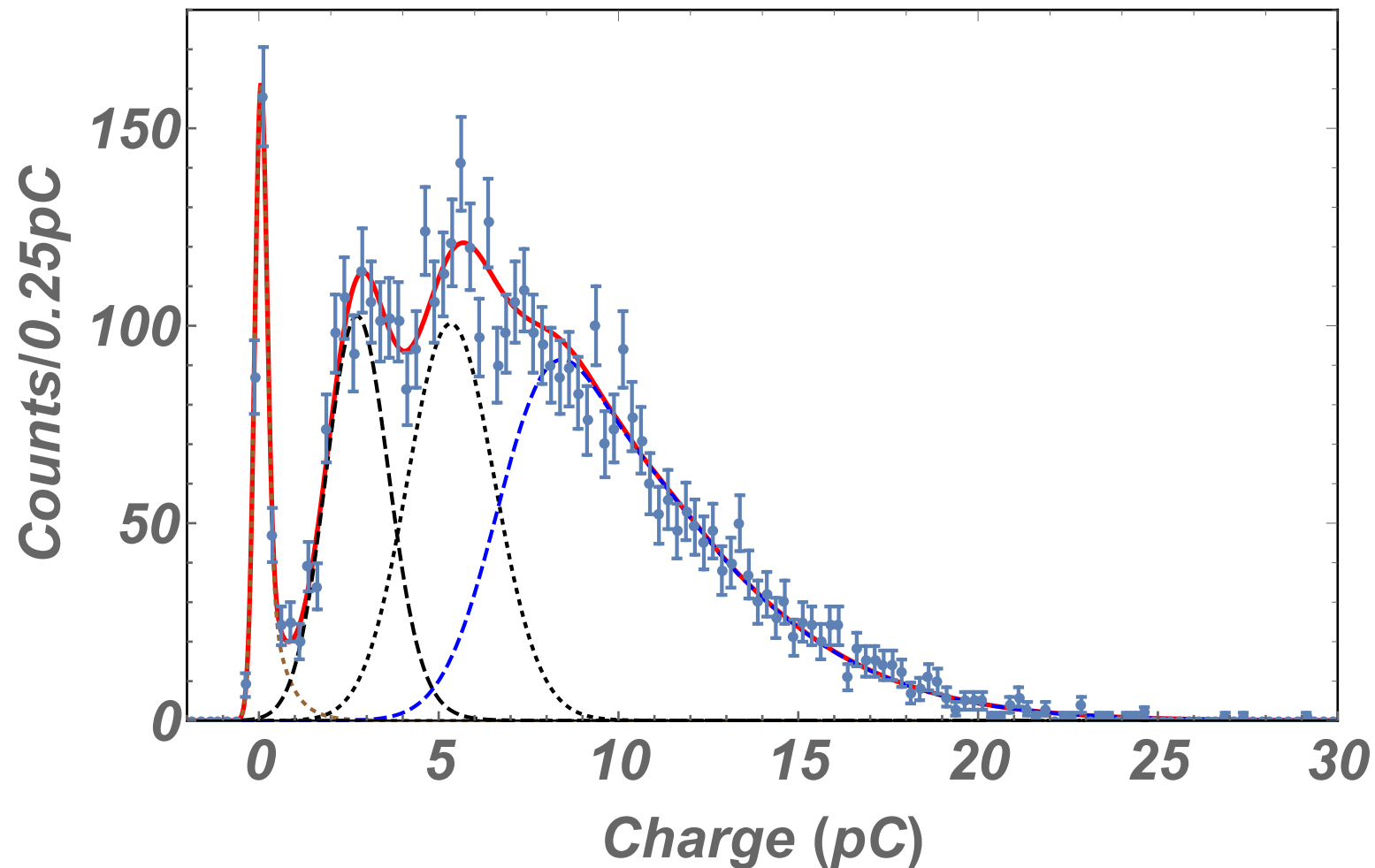
The probability of dark pulses is set to zero, $w=0$. The values of other parameters are set to be $\lambda=5$ (mean photo-electrons), $q_0=1$ (baseline shift), $\sigma_0=0.2$ (baseline fluctuation), $\mu=5$ (mean gain), $\sigma=0.5, 1, 2$ (gain fluctuation) for the magenta-dashed, green-dashed, and blue-solid curves, respectively

some data



- *I have provided some data from a HPK R5912-mod 10 stage PMT. It has very low dark rate. An LED was flashed thru a fiber at the PMT.*
- *Data was taken with a scope and so the pedestal noise is very low also.*
- *There has been no selection of data. 5000 pulses were integrated in a fixed time interval and the LED pulse charge plotted with no cuts.*
- *This PMT is B10-1 at 1430V (left), and 1460V (right)*
- *The red curve is not a fit, I just guessed at the parameters.*
- *Homework: fit the 6 spectra I have provided.*
- *<https://www.phy.bnl.gov/~diwan/software/pmt-spec-code-and-data.tar>*

A detailed fit to data.



The data contains 5000 pulses. The 7 parameter fit resulted in a reduced $\chi^2/\text{DOF}=1.08$ for 77 DOF. The parameters extracted were the baseline shift $q_0=0.044\pm0.031\text{pC}$, the baseline fluctuation $\sigma_0=0.169\pm0.018\text{pC}$, the dark rate probability $w=0.30\pm0.12$, the dark rate exponential parameter $c_0=2.1\pm2.1\text{pC}^{-1}$, the mean gain $\mu=2.59\pm0.06\text{pC}$, the gain fluctuation $\sigma=0.826\pm0.057\text{pC}$ and mean number of photo-electrons $\lambda=2.69\pm0.10$. The figure shows the best fit curve (red) as well as individual components of the spectrum: the charge spectrum for no photo-electron emission (brown dashed) shows a small dark rate component as a tail on the positive side; the single photo-electron spectrum (black dashed), a two photo-electron spectrum (black dotted), and greater than two photo-electrons (blue dashed) are shown separately

conclusion.

- ***We derived the full expression for the charge spectrum from a typical photo-multiplier.***
- ***The expression has parameters for the pedestal, width of the pedestal, the dark current, and the signal.***
- ***The method for deriving the expression is very general, and can be applied to any detector system with appropriate changes.***
- ***The expression can be used for a full fit to an experimental spectrum. It is important to know the stable conditions under which data was obtained.***
- ***I wrote a paper last year with more detail. <https://arxiv.org/abs/1909.05373>***